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Approximating Paley–Wiener Functions by Smoothed Step Functions

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A function f with compactly supported Fourier transform can be approximated by a step function a which coincides with f at regularly spaced points sk, $k \in \mathbb{Z}$. For suitable s, the functions f and a have the same L^2 norm. By modifying a so that its Fourier transform shares the same compact support as that of f, an analytic function is obtained which approximates f, the accuracy depending on s.

1. INTRODUCTION

A function f belonging to L^2 and with Fourier transform $\hat{f}$ supported on a closed interval is called a Paley–Wiener function [8]. The signals occurring in modern communications, such as radio and television, are modelled in a natural way by Paley–Wiener functions (but with some paradoxes which need not concern us here; see [12, 14]). In order to reconstruct such signals from their sampled values $(f(sk))$, $k \in \mathbb{Z}$, a standard practice is to construct a step function from f, then to smooth the step function by restricting its Fourier transform.* Our purpose here is to analyse the error involved in this type of approximation in a general setting which includes multidimensional sampling. The emphasis however is on the case of the one-dimensional or single variable multiband signal, where the spectrum (Fourier transform) is supported on a union of intervals, and on a connection with a local average (Theorem 1.2 below).

*There is a nice discussion from an engineering standpoint of the reconstruction of lowpass signals in "Principles of pulse code modulation," K. W. CATTERMOLE, Iliffe Books Ltd., London, (1969).

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We consider the class of L^2 functions $f\colon \mathbb{R} \to \mathbb{R}$ with Fourier transform $\hat{f}$ vanishing outside a bounded set A; thus the closure of A contains the support of $\hat{f}$. This class will be denoted by PW_A. For suitable positive numbers s depending on A, the functions f have a natural step-function approximation given by horizontal segments of length s with midpoints at the points $(sk, f(sk))$, $k \in \mathbb{Z}$. By putting the Fourier transform to zero outside A, an analytic function is obtained. The question of how well this function approximates the original function f is answered using a generalisation of the Whittaker–Kotel'nikov–Shannon sampling theorem.

The following notation will be used. The Lebesgue measure of the set A will be denoted by $\mu(A)$ and $\|f\|$ will denote the usual L^2 norm unless indicated otherwise by a subscript. The Fourier transform will be normalised so that $\hat{f}$ is given by

$$\hat{f}(x) = \int_{\mathbb{R}} f(t) e^{-2\pi i x t}\, dt,$$

and $\mathscr{F}$ will denote the Fourier transform operator with $\mathscr{F}^{-1}$ the inverse operator.

Paley–Wiener functions f with Fourier transform vanishing outside $(-\omega, \omega)$ can be represented by Whittaker's Cardinal series in terms of values at the points $k/2\omega$ as

$$\begin{aligned} f(x) &= \frac{1}{2\omega} \sum_{k \in \mathbb{Z}} f\left(\frac{k}{2\omega}\right) \frac{\sin 2\pi\omega(x - k/2\omega)}{\pi(x - k/2\omega)} \\ &= \sum_{k \in \mathbb{Z}} f\left(\frac{k}{2\omega}\right) \operatorname{sinc} 2\pi\omega\left(x - \frac{k}{2\omega}\right), \end{aligned} \tag{1}$$

where the sinc function is defined by

$$\operatorname{sinc} x = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0, \\ 1, & x = 0, \end{cases}$$

and the convergence is uniform. This result [3, 9, 15] is also known as the Whittaker–Kotelnikov–Shannon sampling theorem. It has been extended from the case where the Fourier transform vanishes outside an interval symmetric about the origin to that where it vanishes outisde a measurable set A, say, which for some positive real number s satisfies a "disjoint translates" condition that

$$A \cap \left(A + \frac{k}{s} \right) = \varnothing$$

for each non-zero k [4, 13]. This is equivalent to the "difference set" condition

$$D(A) \cap \frac{1}{s}\mathbb{Z} = \{0\}, \tag{2}$$

where $D(A) = \{u - u' \colon u, u' \in A\}$. If this condition holds for A and s, and if the Fourier transform $\hat{f}$ of f vanishes outside A, then

$$f(x) = s \sum_{k \in \mathbb{Z}} f(sk) \int_A e^{2\pi i u(x - sk)}\, du = s \sum_{k \in \mathbb{Z}} f(sk) \check{\chi}_A(x - sk), \tag{3}$$

where the convergence is uniform [4]. The difference set condition implies that $1/s \geq \mu(A)$. This is important as by a result of Landau [11], in order to reconstruct a function f precisely (and stably), the sampling rate (here $1/s$) must be at least the measure of the support of the Fourier transform $\hat{f}$.

The step-function a which forms the first part of the approximation to the function f is given by

$$a(x) = \sum_{k \in \mathbb{Z}} f(sk) \chi_{[-s/2, s/2)}(x - sk) \tag{4}$$

and coincides with f at the midpoint sk of each interval $[s(k - 1/2), s(k + 1/2))$, $k \in \mathbb{Z}$. For convenience we take the sample $f(sk)$ to be at the center of the horizontal segment, rather than at the left edge (as is the practice in electronics). The only difference is an irrelevant phase shift of $s/2$.

Let us denote the operation of constructing a by $\mathscr{A}$, thus

$$\mathscr{A} \colon (f(sk)) \mapsto a.$$

In fact, $\mathscr{A} \colon \ell^2 \mapsto L^2(\mathbb{R})$. The following lemma shows more: it is an isometry.

1.1. Lemma. *Let $f \in PW_A$. Then*

$$\|a\| = s^{1/2} \| f(sk) \|_{\ell^2} = \|f\|.$$

Proof. For

$$\int_{\mathbb{R}} |a(x)|^2\,dx = \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} \sum_{k'\in\mathbb{Z}} f(sk)\overline{f(sk')}\chi_{[-s/2,s/2)}(x-sk) \times \chi_{[-s/2,s/2)}(x-sk')\,dx$$
$$= \sum_{k\in\mathbb{Z}} |f(sk)|^2 \int_{s(k-1/2)}^{s(k+1/2)} dx = s\sum_{k\in\mathbb{Z}} |f(sk)|^2 = \|f\|^2,$$

where, because s satisfies (2), the last equality holds by [4, Thm. 2]. ∎

Of course a is not continuous and the support of its Fourier transform $\hat{a}$ is the whole real line. In order to smooth a, subject it to the "band-limiting" operator $\mathscr{B} = \mathscr{F}^{-1}\chi_A\mathscr{F}$, where here χ_A is taken to mean multiplication by $\chi_A(u)$, i.e., $(\chi_A\mathscr{F})(f)(u) = \chi_A(u)(\mathscr{F}f)(u)$. Let

$$\mathscr{B}: a \mapsto r;$$

then we can define

$$\mathscr{R} = \mathscr{B}\mathscr{A}: (f(sk)) \mapsto r.$$

This is not the only possible choice for a smooth approximation to the original function f (for instance, smooth bump functions or splines could be used to remove the discontinuities from a), but it is a natural one since the support of the Fourier transforms of the smoothed step-function r and the original function f are the same. Now the function r is given by $r = \mathscr{B}(a) = \mathscr{F}^{-1}\chi_A\mathscr{F}a = (\chi_A\hat{a})\hat{}$ and we will show that

$$r(x) = s\int_A \sum_{k\in\mathbb{Z}} f(sk)\operatorname{sinc}(\pi su)e^{2\pi i(x-sk)u}\,du. \tag{5}$$

Surprisingly, this process of deriving r from f (via the samples of f) has an alternative formulation in terms of a "moving average" operator $\mathscr{M}$ which does not involve Fourier transformation.

1.2. THEOREM. *Let*

$$m(x) = (\mathscr{M}f)(x) = \frac{1}{s}\int_{x-s/2}^{x+s/2} f(v)\,dv = \frac{1}{s}\int_{\mathbb{R}} f(v)\chi_{[-s/2,s/2]}(x-v)\,dv.$$

Then $m(x) = r(x)$.

Proof. From [4, Thm. 5],

$$\lim_{N\to\infty}\int_A\left|\hat{f}(u)-s\sum_{k=-N}^{N}f(sk)e^{2\pi isku}\right|^2du=0,$$

so that

$$\lim_{N\to\infty}\int_A\left(\hat{f}(u)-s\sum_{k=-N}^{N}f(sk)e^{2\pi isku}\right)\operatorname{sinc}(\pi su)e^{2\pi ixu}\,du$$

$$\le\left[\lim_{N\to\infty}\int_A\left|\hat{f}(u)-s\sum_{k=-N}^{N}f(sk)e^{2\pi isku}\right|^2du\right]^{1/2}\left[\int_A|\operatorname{sinc}(\pi su)|^2\,du\right]^{1/2}$$

$$=0.$$

Hence,

$$\int_A s\sum_{k\in\mathbb{Z}}f(sk)e^{2\pi isk}\operatorname{sinc}(\pi su)e^{2\pi ixu}\,du=\int_A\hat{f}(u)\operatorname{sinc}(\pi su)e^{2\pi ixu}\,du$$

$$=\frac{1}{s}\int_{\mathbb{R}}f(v)\chi_{[-s/2,s/2]}(x-v)\,dv$$

by Parseval's theorem. ∎

The connection between sampling theory and local averages has also been noted independently by Feichtinger and Gröchenig in a somewhat different setting. They study irregular sampling for functions with A an interval using frames and give an iterative procedure for reconstruction from local averages [6, Thm. 7].

The accuracy of the approximating function r is given by the error $\varepsilon=f-r$ where

$$\varepsilon(x)=s\sum_{k\in\mathbb{Z}}f(sk)\int_A(1-\operatorname{sinc}\pi su)e^{2\pi iu(x-sk)}\,du. \tag{6}$$

The error ε will now be estimated in L^2-norm and pointwise.

2. The Norm of the Error ε

First the norm of the error is expressed in terms of the attenuation of the spectrum.

2.1. Theorem. *When s satisfies* (2), *the norm $\|\varepsilon\|$ of the error ε satisfies*

$$\|\varepsilon\|^2 = \int_A \left(1 - \frac{1}{s}\hat{\chi}_{[-s/2, s/2)}(u)\right)^2 |\hat{f}(u)|^2 \, du = \int_A \left(1 - \frac{\sin \pi s u}{\pi s u}\right)^2 |\hat{f}(u)|^2 \, du. \tag{7}$$

Proof. Since s satisfies (2), $f(x)$ has representation (3). Next, let

$$f_N(x) = s \sum_{k=-N}^{N} f(sk) \check{\chi}_A(x - sk) = s \sum_{k=-N}^{N} f(sk) \int_A e^{2\pi i (x - sk) u} \, du,$$

so that for each x, $f_N(x) \to f(x)$ as $N \to \infty$, and so that

$$\hat{f}_N(u) = s \sum_{k=-N}^{N} f(sk) e^{-2\pi i s k u} \chi_A(u).$$

Similarly, let

$$a_N(x) = \sum_{k=-N}^{N} f(sk) \chi_{[-s/2, s/2)}(x - sk),$$

so that

$$\begin{aligned} \hat{a}_N(u) &= \sum_{k=-N}^{N} f(sk) \int_{s(k-1/2)}^{s(k+1/2)} e^{-2\pi i u x} \, dx \\ &= \hat{\chi}_{[-s/2, s/2)}(u) \sum_{k=-N}^{N} f(sk) e^{-2\pi i s k u} \\ &= s \operatorname{sinc}(\pi s u) \sum_{k=-N}^{N} f(sk) e^{-2\pi i s k u}. \end{aligned}$$

The Fourier transform $\hat{r}_N$ of the N-th partial sum r_N of r is given by

$$\hat{r}_N(u) = \hat{a}_N(u)\chi_A(u) = s\,\mathrm{sinc}(\pi su) \sum_{k=-N}^{N} f(sk)e^{-2\pi isku}\chi_A(u)$$

$$= \mathrm{sinc}(\pi su)\hat{f}_N(u),$$

whence

$$r_N(x) = s\int_A \sum_{K=-N}^{N} f(sk)\mathrm{sinc}(\pi\, su)e^{2\pi i(x-sk)u}du$$

and (5) follows. $f_N - r_N = \varepsilon_N \to \varepsilon$ as $N \to \infty$, and so by Parseval's formula. Also

$$\|\varepsilon_N(u)\|^2 = \|f_N - r_N\|^2 = \|\hat{f}_N - \hat{r}_N\|^2$$
$$= \int_A \left|\hat{f}_N(u) - \hat{r}_N(u)\right|^2 du = \int_A \left|\hat{f}_N(u)\right|^2 (1 - \mathrm{sinc}\,\pi su)^2\, du.$$

Finally,

$$\left|\|\varepsilon\|^2 - \int_A \left|\hat{f}(u)\right|^2 (1 - \mathrm{sinc}\,\pi su)^2\, du\right|$$
$$\leq \left|\|\varepsilon\|^2 - \|\varepsilon_N\|^2\right| + \left|\int_A \left(\left|\hat{f}_N(u)\right|^2 - \left|\hat{f}(u)\right|^2\right)(1 - \mathrm{sinc}\,\pi su)^2\, du\right|,$$

and the result follows on letting $N \to \infty$. ∎

As a referee has pointed out, this theorem could also be proved using Poisson's summation formula when A is an interval.

2.2. Corollary. *As $s \to 0$,*

$$\|\varepsilon\| = Cs^2 + O(s^4),$$

where

$$C = \frac{\pi^2}{6}\left(\int_A u^4 \left|\hat{f}(u)\right|^2 du\right)^{1/2}.$$

Proof. Since A is a bounded set of $\mathbb{R}$, (2) will be satisfied and hence the above theorem will hold when s sufficiently small. Also, for small s,

$$1 - \mathrm{sinc}(\pi su) = \frac{(\pi su)^2}{6} + O(s^4),$$

and substituting for the integrand in (7) gives the required result. ∎

A more explicit asymptotic expression is obtained for functions f with $A = (-\omega, \omega)$ and $s \leq 1/2\omega$ (such functions are called *lowpass* signals). The constant $1/5$ is chosen for simplicity.

2.3. COROLLARY. *When* $A = (-\omega, \omega)$ *and* $s \leq 1/2\omega$,

$$\|\varepsilon\| < \frac{\pi^2 s^2 \omega^2}{5}\|f\| = \omega^2 \|f\| O(s^2).$$

Proof. We note that

$$1 - \operatorname{sinc}(v) = \frac{v^2}{3!}\left(1 - \frac{v^2}{4 \cdot 5} + \frac{v^4}{4 \cdot 5 \cdot 6 \cdot 7} + \cdots\right) < \frac{v^2}{6} \frac{1}{1 - (v/4)^2}.$$

Since $v = \pi s u$, then $|v| \leq \pi s \omega \leq \pi/2$ and $1/(1 - (v/4)^2) < 1.2$. Hence, from Theorem 2.1,

$$\|\varepsilon\| < \int_{-\omega}^{\omega} \frac{\pi^4 s^4 u^4}{36} 1.2^2 |\hat{f}(u)|^2 \, du = \frac{\pi^4 s^4 \omega^4}{36} 1.2^2 \int_{-\omega}^{\omega} |\hat{f}(u)|^2 \, du,$$

from which the corollary follows. ∎

Thus our estimates are quadratic in s, and for the lowpass case quadratic in ω (when also $s\omega \leq 1/2$). In [7], Gröchenig considers the problem of irregular sampling in the lowpass case and obtains an estimate which is linear in δ, the supremum of the distance between consecutive sampling points. In the case of regular sampling this estimate is linear in s and ω. However application of his iteration technique then increases the order of δ at each step.

It also follows from these results that "oversampling" (sampling at rates which considerably exceed 2ω) is very effective in reducing the error arising from the processing. For example, eightfold oversampling (as advertised on some compact disc players) reduces the energy of the error by a factor of about 2^{-6}, i.e., by about 98.4%.

When the set A outside which $\hat{f}$ vanishes is not an interval about the origin, the error estimates are not necessarily so good.

2.4. COROLLARY. *Suppose* $A = (-(c + 1)\omega, -c\omega) \cup (c\omega, (c + 1)\omega)$ *and that* $c > 1/(\pi s \omega)$. *Then*

$$\|\varepsilon\| \geq \left(1 - \frac{1}{\pi s c \omega}\right)\|f\|.$$

Proof. By definition,

$$\|\varepsilon\|^2 = 2\int_{c\omega}^{(c+1)\omega}\left(1 - \frac{\sin \pi s u}{\pi s u}\right)^2 |\hat{f}(u)|^2 \, du$$
$$\geq \left(1 - \frac{1}{\pi s c \omega}\right)^2 \|f\|^2. \quad ∎$$

Thus if c is large (i.e, the signal has high frequency content only) the error is going to be large unless the sampling interval s is very small. In practice engineers do not reconstruct this type of signal (called *bandpass*) by this technique (see also [1]).

A referee has pointed out that Theorem 2.1 can be extended to the multidimensional case. In fact, the theorem can be placed into the abstract setting of locally compact abelian groups, which includes the multidimensional case and which has the advantage of clarifying the relation between the measure and the sampling interval. This method of reconstruction is however used principally in audio-signal reconstruction and therefore this generalisation is unlikely to be of practical use.

Let G be a locally compact abelian group with dual Γ, let (x, u) denote a character and let the Haar measure on Γ be normalised so that the Fourier inversion formula holds. Let Λ be a discrete subgroup of Γ with Γ/Λ compact and let H be the annihilator of Λ or equivalently the dual of Γ/Λ. Then H is discrete and G/H is compact. Let Ω be a complete set of coset representatives of Γ/Λ in Γ and let $\kappa = m_\Gamma(\Omega)$ be the Haar measure of Ω in Γ. Let I be a complete set of coset representatives of G/H in G. Then it can be shown that the Haar measure of I in G is κ^{-1}.

In the one-dimensional case G and its dual Γ are the real line with the Haar measure of G Lebesgue measure; the character $(x, u) = e^{2\pi i x u}$, $\Lambda = s^{-1}\mathbb{Z}$, $H = s\mathbb{Z}$, and $I = [-s/2, s/2)$. Note that $\mu(I) = s$.

2.5. THEOREM. *Let $f \in L^2(G)$ with $\hat{f} = 0$ for $u \notin A \subseteq \Omega$. The error in approximating f by the smoothed step-function r is*

$$\|\varepsilon\| = \left\| \hat{f}(1 - \kappa \hat{\chi}_I) \right\|.$$

Proof. By [5],

$$f(x) = \kappa^{-1} \sum_{h \in H} f(h) \check{\chi}_A(x - h),$$

with

$$\hat{f}(u) = \kappa^{-1} \sum_{h \in H} f(h)(-h, u) \chi_A(u).$$

In this setting, the step-function approximation to f is given by

$$a(x) = \sum_{h \in H} f(h) \chi_I(x - h),$$

with its Fourier transform given by

$$\hat{a}(u) = \sum_{h \in H} f(h)(-h, u)\hat{\chi}_I(u).$$

Therefore, omitting the details of convergence, the Fourier transform of the reconstructed function r is given by

$$\begin{aligned}\hat{r}(u) &= \hat{\chi}_I(u) \sum_{h \in H} f(h)(-h, u)\chi_A(u) \\ &= \kappa\hat{\chi}_I(u)\hat{f}(u).\end{aligned}$$

Hence,

$$\|\varepsilon\| = \|\hat{f} - \hat{r}\| = \|\hat{f} - \kappa\hat{\chi}_I\hat{f}\| = \left\|\hat{f}(1 - \kappa\hat{\chi}_I)\right\|$$

as required. ∎

Theorem 1.2 also extends naturally to locally compact abelian groups. With the setting of Theorem 2.5, it can be shown that for each $x \in G$.

$$r(x) = \int_\Gamma \hat{f}(u)X_I(u)(x, u)\,du = \kappa\int_{I+x} f(w)\,dw$$

3. Pointwise Estimates for ε

Expression (6) for $\varepsilon(x)$ gives a simple estimate under the l^1 assumption that

$$\sum_{k \in \mathbb{Z}} |f(sk)| < \infty,$$

since

$$|\varepsilon(x)| \leq 1.3\mu(A)s \sum_{k \in \mathbb{Z}} |f(sk)| = O(s).$$

The following theorem gives a more precise result.

3.1. Theorem. *For each* x,

$$|\varepsilon(x)| \leq \|f\|\left(\int_A (1 - \operatorname{sinc} \pi su)^2\,du\right)^{1/2}. \tag{8}$$

Proof. From (6),

$$\varepsilon(x) = f(x) - r(x) = s\sum_{k \in \mathbb{Z}} f(sk)\int_A (1 - \operatorname{sinc} \pi su)e^{2\pi iu(x-sk)}\,du.$$

Now, write

$$g(y) = \int_A (1 - \operatorname{sinc} \pi su) e^{2\pi i u y}\, du = \int_{\mathbb{R}} \chi_A(u)(1 - \operatorname{sinc} \pi su) e^{2\pi i u y}\, du.$$

Then $g \in L^2(\mathbb{R})$, $\hat{g}(u) = \chi_A(u)(1 - \operatorname{sinc} \pi su)$,

$$\|g\|^2 = \|\hat{g}\|^2 = \int_A |\hat{g}(u)|^2\, du = \int_A (1 - \operatorname{sinc} \pi su)^2\, du.$$

Moreover, from [4, Thm. 4],

$$\begin{aligned} \varepsilon(x) &= s \sum_{k \in \mathbb{Z}} f(sk) g(x - sk) = (f * g)(x) \\ &= \int_A \hat{f}(u) \hat{g}(u) e^{2\pi i x u}\, du, \end{aligned}$$

and so by Cauchy's inequality,

$$\begin{aligned} |\varepsilon(x)| &\le \left(\int_A |\hat{f}(u)|^2\, du \right)^{1/2} \left(\int_A |\hat{g}(u)|^2\, du \right)^{1/2} \\ &= \|f\| \left(\int_A (1 - \operatorname{sinc} \pi su)^2\, du \right)^{1/2} \end{aligned}$$

as claimed. ∎

There is a more explicit estimate for Paley–Wiener functions f where $\hat{f}(u) = 0$ when $|u| > \omega$, i.e., when f is a lowpass signal.

3.2. Corollary. *When $\hat{f}$ vanishes outside $(-\omega, \omega)$, if $s \le 1/2\omega$*

$$|\varepsilon(x)| < \frac{\sqrt{2}\, \pi^2 s^2 \omega^{5/2}}{5\sqrt{5}} \|f\|.$$

Proof. The proof follows from (8) following similar lines to the proof of Corollary 2.3. ∎

4. An Application

The analysis in Theorem 2.1 suggests a modification of the standard method for digital to analogue conversion which gives better signal reconstruction [2]. The improvement is achieved by altering the profile of the step-function a as follows. The amplitude of each step is multiplied by a constant factor $\beta \geq 1$ but only held at this value for a fraction $0 < \sigma \leq 1$ of the sampling interval s, and a is set to zero for the remainder of the sampling interval. Therefore, instead of (4), the step-function approximation to f is given by

$$a(t) = \sum_{k \in \mathbb{Z}} \beta f(sk) \chi_{[0, \sigma s)}(t - sk).$$

The norm of the error (for a fixed sampling rate) is then given by

$$\|\varepsilon\|^2 = \int_A \left(1 - \beta\sigma \frac{\sin \pi s u}{\pi s u}\right)^2 |\hat{f}(u)|^2 \, du.$$

It is natural to choose $\beta = 1/\sigma$, then the error is essentially $O(\sigma^2)$. A detailed analysis of this technique and some examples are given in [2]. A suitable application for this improved digital-to-analogue conversion method is a speech compression technique which employs multiband sampling [10]. For lowpass signals, quite different and highly effective methods of improving signal reconstruction have been developed recently.

5. Conclusion

Both types of error estimate considered depend quadratically on the sampling interval s. However, differences arise in the dependence of the constants on the set A. This is most clearly seen in the case of lowpass functions where $A = (-\omega, \omega)$. The constant for the norm estimate depends on ω^2, whereas for the pointwise error it depends on $\omega^{5/2}$.

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